Detecting areas with synchronous temporal dynamics

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1 Model and estimation procedure

1.1 Goal

We write Z_{stk} for the kth observations, year t, site s and $z_{st} = \sum_k Z_{stk}$. Our goal is to estimate regions R such that

$$Z_{stk} \sim \text{Poisson}(\exp(\theta_s + f(x_s, t))) \quad \text{with } f(x,t) \approx \sum_R \rho_R(t) \mathbf{1}_{x \in R}. \tag{1}$$

In other words, we try to estimate f with the a priori that

- for each year t the map $x \to f(x,t)$ is piecewise constant
- the boundary of the regions where $x \to f(x,t)$ is constant are the same for all year t.

The main difficulty is to detect the regions R.

1.2 Estimation procedure

Let G be a graph and write $V(s)$ for the set of the neighbors of s in G. The estimators $\widehat{\theta}$ and $\widehat{f}$ are defined as minimizers of

$$\mathcal{L}(\theta, f) + \alpha \text{pen}(f) := \sum_{s,t} [e^{\theta_s + f_{st}} - z_{st}(\theta_s + f_{st})] + \alpha \sum_{s \overset{G}{\sim} u} \|f_{s.} - f_{u.}\| / D_{su}$$

avec les conditions d'identifiabilité: $f_{s1} = 0$ pour tout s. On choisira typiquement $D_{su} = 1/|V(s)| + 1/|V(u)|$.

2 Optimization algorithm

On a à minimiser

$$\mathcal{L}(\theta, f) + \alpha \text{pen}(f) := \sum_{s,t} [e^{\theta_s + f_{st}} - z_{st}(\theta_s + f_{st})] + \alpha \sum_{s \overset{G}{\sim} u} \|f_{s.} - f_{u.}\| / D_{su}$$

avec les conditions d'identifiabilité: $f_{s1} = 0$ pour tout s. On a

$$\mathcal{L}(\theta, f) + \alpha \text{pen}(f) = \sum_{s,t} [e^{\theta_s + f_{st}} - z_{st}(\theta_s + f_{st})] + \alpha \sum_{s \overset{G}{\sim} u} \max_{\|\phi_{su}\| \le 1} \langle \phi_{su}, f_{s.} - f_{u.} \rangle / D_{su}$$

avec $\phi_{su} \in \mathbf{R}^T$.

On introduit

$$F(\theta, f, \phi) = \sum_{s,t} [e^{\theta_s + f_{st}} - z_{st}(\theta_s + f_{st})] + \alpha \sum_{s<u} \mathbf{1}_{s \overset{G}{\sim} u} \langle \phi_{su}, f_{s.} - f_{u.} \rangle / D_{su}.$$

et

$$\mathcal{L}(\theta, f) + \alpha \text{pen}(f) = \max_{\max_{s<u} \|\phi_{su}\| \leq 1} F(\theta, f, \phi).$$

Mise en oeuvre:

Itérer jusqu'à convergence:

1. descente de gradient en θ:
   $\theta \leftarrow \theta - h \nabla_\theta F$
2. descente de gradient en f avec condition $f[\ ,1] = 0$
   $f[\ ,-1] \leftarrow f[\ ,-1] - h' \nabla_{f[\ ,-1]} F$
3. montée de gradient en ϕ
   $\phi_{su} \leftarrow \phi_{su} + h'' \nabla_{\phi_{su}} F$
4. $\phi_{su} \leftarrow \phi_{su} / \max(1, \|\phi_{su}\|)$

Return(θ, f)

Gradient en θ:

On a

$$\mathcal{L}(\theta, f) = \sum_s \left[e^{\theta_s} \sum_t e^{f_{st}} - \theta_s \sum_t z_{st} \right] + \dots$$

Donc

$$\partial_{\theta_s} F = e^{\theta_s} \sum_t e^{f_{st}} - \sum_t z_{st}$$

Gradient en f:

on note $\phi_{su} = -\phi_{us}$ pour $s > u$

$$\partial_{f_{st}} F = e^{\theta_s} e^{f_{st}} - z_{st} + \alpha \sum_{u \in V(s)} [\phi_{su}]_t / D_{su}$$

Gradient en λ:

pour $s < u$ avec $s \sim u$

$$\nabla_{\phi_{su}} F = \alpha (f_{s.} - f_{u.}) / D_{su}$$